[11] Fourier Series

[11.1] Fourier Series :→

Fourier series comes from the periodic fun^n f(x) in terms of cosine and sine fun^{n}s.
These series are trigonometric series i.e

$a_0 + a_1 \cos x + a_2 \cos 2x + \dots + b_1 \sin x + b_2 \sin 2x + \dots$

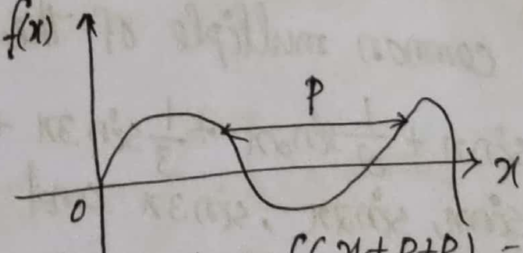
These co-efficients are determined by Euler formulas.

Periodic fun^n :→ Trigonometric series.

Period :→ A same thing will occur continuously after a certain interval. it is called as period.

Periodic Fun^n :→
A fun^n f(x) is called periodic if it is defined for all real x, if there is some positive no. P such that, $f(x+P) = f(x), \forall x$
where 'P' is a period of f(x).

Graph of Periodic Function :→



If we write $f(x+2P) = f(x+P+P) = f(x)$
similarly $f(x+nP) = f(x), \forall x$
Hence 2P, 3P, 4P... are also periods of f(x).
If a periodic fun^n f(x) has a smallest period P(>0), then this is called as fundamental period of f(x).

Ex-1 1. cos x, sin x, sec x, cosec x are periodic fun^n with period 2π.

2. term, etc. are periodic with period π.

Note-1 If p is the period of $f(x)$, then 'np' is also period of 'f' for any integer 'n'

i.e $f(x+np) = f(x)$ $(n \neq 0)$

Ex:→ $\cos(x+4\pi) = \cos(x+2\cdot 2\pi) = \cos((x+2\pi)+2\pi)$
$= \cos(x+2\pi) = \cos x$

Note-2 The funⁿ $h(x) = a f(x) + b g(x)$ has period 'p' if $f(x)$ & $g(x)$ have period p. Here a & b are constants.

Ex:→ $h(x) = a\cos x + b\sin x$
$h(x+2\pi) = a\cos(x+2\pi) + b\sin(x+2\pi) = a\cos x + b\sin x$
$= h(x)$

Note-3 If $f(x)$ is a periodic funⁿ of period p, then $f(ax)$ with $a \neq 0$ is a periodic funⁿ of period $\frac{p}{a}$.

Ex $\cos 2x$ has period $\frac{2\pi}{2} = \pi$
$\sin 3x$ has period $\frac{2\pi}{3}$ and so on

~~Ex~~

Note-4 The period of a sum of a no. of periodic funⁿ is the least common multiple of the periods.

Ex $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x$

Note that $\sin x$, $\sin 2x$, $\sin 3x$ and $\sin 4x$ has period $2\pi, \pi, \frac{2\pi}{3}, \frac{\pi}{2}$ resp.

Then the period is 2π which is L.C.M. of these periods.

Note-5 A constant funⁿ is periodic for any positive p.

Euler Formula for the Fourier co-efficient :→

The fourier series for the funⁿ $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——— (1)}$$

Thus, $a_0 = \frac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$

Thus a_0, a_n, b_n are known as fourier co-efficients.

Proof $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

To find a_0 :→

integrate eqn-(1) from α to $\alpha+2\pi$

$$\int_{\alpha}^{\alpha+2\pi} f(x)dx = \int_{\alpha}^{\alpha+2\pi} a_0\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= a_0(\alpha+2\pi-\alpha) + 0 + 0$$

$$= 2a_0\pi$$

$$\Rightarrow \boxed{a_0 = \frac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx} \quad \left(\because \int_{\alpha}^{\alpha+2\pi}\cos nx\,dx = 0 = \int_{\alpha}^{\alpha+2\pi}\sin nx\,dx;\ \text{for } n \neq 0\right)$$

To find a_n :→

Multiply '$\cos nx$' on both sides & integrate eqn (1) from α to $\alpha+2\pi$;

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx = \int_{\alpha}^{\alpha+2\pi} a_0 \cos nx\,dx$$

$$+ \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx$$

$$+ \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$$= a_0 \cdot 0 + a_n \cdot \pi + 0 = a_n\pi$$

$$\boxed{a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cdot\cos nx\,dx.}$$

$$\left(\because \int_{\alpha}^{\alpha+2\pi}\cos^2 nx\,dx = \pi;\ \int_{\alpha}^{\alpha+2\pi}\cos nx\,dx = 0;\ \int_{\alpha}^{\alpha+2\pi}\sin nx\cdot\cos nx\,dx = 0\right)$$

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To find b_n :→

Multipling '$\sin nx$' on both sides & integrate eqn - ①
from α to $\alpha+2\pi$.

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = \int_{\alpha}^{\alpha+2\pi} a_0 \sin nx\,dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + b_n \cdot \pi$$

$$= b_n\pi$$

$$\Rightarrow \boxed{b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cdot\sin nx\,dx} \quad \left(\because \int_{\alpha}^{\alpha+2\pi}\sin nx\,dx = 0,\ \int_{\alpha}^{\alpha+2\pi}\sin^2 nx\,dx = \pi,\ \int_{\alpha}^{\alpha+2\pi}\sin nx\cdot\cos nx\,dx = 0\right)$$

This is known as Euler formula .

Note

① $\int_{-\pi}^{\pi} \cos mx\cdot\cos nx\,dx = 0 \quad (m\neq n)$

(2) $\int_{-\pi}^{\pi} \sin mx\cdot\sin nx\,dx = 0 \quad (m\neq n)$

(3) $\int_{-\pi}^{\pi} \cos mx\cdot\sin nx\,dx = 0 \quad (m=n)$

(4) $\sin(-\theta) = -\sin\theta$ (odd funn)

(5) $\cos(-\theta) = \cos\theta$ (even funn)

(6) $\sin n\pi = 0$

Let n & m be integers , $n\neq 0$, $m\neq 0$:—

For $m\neq n$

(1) $\int_{\alpha}^{\alpha+2\pi} \cos mx\cdot\cos nx\,dx = 0$

(2) $\int_{\alpha}^{\alpha+2\pi} \sin mx\cdot\sin nx\,dx = 0$

(3) $\int_{\alpha}^{\alpha+2\pi} \sin mx\cdot\cos nx\,dx = 0$

(4) $\int_{\alpha}^{\alpha+2\pi} \cos mx\,dx = 0$

(5) $\int_{\alpha}^{\alpha+2\pi} \sin nx\,dx = 0$

For $m=n$:→

(1) $\int_{\alpha}^{\alpha+2\pi} \cos mx\cdot\cos nx\,dx = \int_{\alpha}^{\alpha+2\pi}\cos^2 mx\,dx = \pi$

(2) $\int_{\alpha}^{\alpha+2\pi} \sin^2 mx\,dx = \pi$

(3) $\int_{\alpha}^{\alpha+2\pi} \cos mx\cdot\sin nx\,dx = 0$

Exercise- 11.1

No-2 (i) $f(x) = \cos x$ — ①
$f(x+P) = \cos(x+P)$ — ②
we apply $f(x+P) = f(x)$
so $\cos(x+P) = \cos x$
$\Rightarrow \cos(x+P) - \cos x = 0$
$\Rightarrow 2\cdot\sin\left(\frac{2x+P}{2}\right)\cdot\sin\left(-\frac{P}{2}\right) = 0$ [$\because -\sin\theta = \sin(-\theta)$]
$\Rightarrow -2\cdot\sin\frac{2x+P}{2}\cdot\sin\left(\frac{P}{2}\right) = 0$
$\Rightarrow \sin\left(\frac{P}{2}\right) = 0$; $\frac{P}{2} = \pi$
$\Rightarrow \boxed{P = 2\pi}$

(1) $\cos A - \cos B = 2\cdot\sin\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{B-A}{2}\right)$
(2) $\cos A + \cos B = 2\cdot\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$
(3) $\sin A + \sin B = 2\cdot\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
(4) $\sin A - \sin B = 2\cdot\cos\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)$

$x - x\left(\frac{-P}{2}\right)$

$\sin\left(\frac{P}{2}\right) = 0$
$\Rightarrow \sin\left(\frac{P}{2}\right) = \sin\pi$
$\Rightarrow \frac{P}{2} = \pi$

$2\cdot\sin A\cdot\cos B$
$= \sin(A+B) + \sin(A-B)$

(ii) $f(x) = \cos\pi x$
$f(x+P) = \cos\pi(x+P)$
we apply for P; $f(x+P) = f(x)$
$\therefore \cos\pi(x+P) = \cos\pi x$
$\Rightarrow \cos\pi(x+P) - \cos\pi x = 0$
$\Rightarrow 2\cdot\sin\left(\frac{2\pi x+P}{2}\right)\cdot\sin\left(\frac{-\pi P}{2}\right) = 0$
$\Rightarrow -2\sin\left(\pi x+\frac{P}{2}\right)\cdot\sin\left(\frac{\pi P}{2}\right) = 0$
$\sin\left(\frac{\pi P}{2}\right) = 0 \Rightarrow \frac{\pi P}{2} = \pi$ $\Rightarrow \boxed{P = 2}$

(iii) $f(x) = \sin 2\pi x$
$f(x+P) = \sin 2\pi(x+P)$
we apply $f(x+P) = f(x)$
so $\sin 2\pi x = \sin 2\pi(x+P)$
$\Rightarrow \sin 2\pi(x+P) - \sin 2\pi x = 0$
$\Rightarrow 2\cdot\cos(2\pi x+\pi P)\cdot\sin(\pi P) = 0$
$\Rightarrow \sin\pi P = 0$
$\Rightarrow \pi P = \pi$
$\Rightarrow \boxed{P = 1}$

No-2 (ii) $f(x) = \sin nx$

$f(x+P) = \sin n(x+P)$

we apply for finding 'p',

$f(x+P) = f(x)$

$\therefore \sin n(x+P) = \sin nx$

$\Rightarrow \sin n(x+P) - \sin nx = 0$

$\Rightarrow 2\cos\left(nx + \frac{nP}{2}\right) \cdot \sin\left(-\frac{nP}{2}\right) = 0$

$\Rightarrow -2\cos\left(nx + \frac{nP}{2}\right) \cdot \sin\left(\frac{nP}{2}\right) = 0$

$\Rightarrow \sin\left(\frac{nP}{2}\right) = 0$

$\Rightarrow \frac{nP}{2} = \pi$

$\Rightarrow \boxed{P = \frac{2\pi}{n}}$

(iii) $f(x) = \cos \frac{2\pi n x}{K}$

$f(x+P) = \cos \frac{2\pi n(x+P)}{K}$

$\therefore \cos \frac{2\pi n}{K}(x+P) = \cos \frac{2\pi n}{K} x$

$\Rightarrow \cos \frac{2\pi n}{K}(x+P) - \cos \frac{2\pi n}{K} x = 0$

$\Rightarrow 2\sin\left(\frac{2\pi n x}{K} + \frac{\pi n P}{K}\right) \cdot \sin\left(-\frac{\pi n P}{K}\right) = 0$

$\Rightarrow -2\sin\left(\frac{2\pi n x}{K} + \frac{\pi n P}{K}\right) \cdot \sin\left(\frac{\pi n P}{K}\right) = 0$

$\Rightarrow \sin\left(\frac{\pi n P}{K}\right) = 0 \Rightarrow \frac{\pi n P}{K} = \pi \Rightarrow \boxed{P = \frac{K}{n}}$.

4 $f(x) = \text{const.}$

$f(x+P) = f(x) = \text{const.}$

Hence $f(x)$ is periodic with any period but has no fundamental period.

5 $f(x+P) = f(x)$, $g(x+P) = g(x)$

$h = af + bg$

$h(x+P) = (af+bg)(x+P) = af(x+P) + bg(x+P)$

$= af(x) + bg(x) = h(x)$

Hence $h(x+P) = h(x)$, where P is period.

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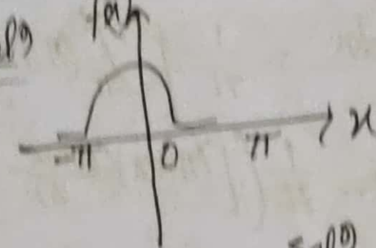
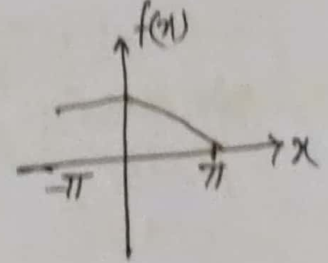
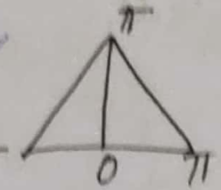
3. $f(x) = e^{-|x|}$ $(-\pi < x < \pi)$ Sol^n



12. $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ \cos\frac{x}{2} & \text{if } 0 < x < \pi \end{cases}$ Sol^n



15. Q.17



Sol^n Here $(-\pi, 0)$ to $(0, \pi)$ then we apply distance formula.

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1} \Rightarrow \frac{\pi}{\pi} = \frac{f(x)}{x + \pi} \Rightarrow f(x) = \pi + x$

Here $(0, \pi)$ to $(\pi, 0)$ then we apply distance formula

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1} \Rightarrow \frac{-\pi}{\pi} = \frac{f(x) - \pi}{x} \Rightarrow f(x) = \pi - x$

$$f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$$

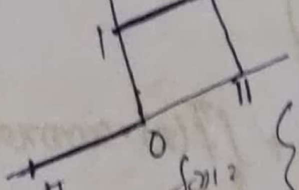
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} (\pi + x)\,dx + \int_{0}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{1}{2\pi}\left(\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{2\pi}\left(\left[0 - \left(\frac{-\pi^2}{2}\right)\right] + \left[\left(\frac{\pi^2}{2}\right) - 0\right]\right)$$

$$= \frac{1}{2\pi}(\pi^2) = \frac{\pi}{2}$$

18.



$f(x) = \begin{cases} 0; & -\pi < x < 0 \\ 1; & 0 < x < \pi \end{cases}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$ $\left[\int uv\,dx = u \int v\,dx - \int \left(\frac{du}{dx} \cdot \int v\,dx\right) dx\right]$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x) \cos nx\,dx + \int_{0}^{\pi} (\pi - x) \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left(\left[(\pi + x)\frac{\sin nx}{n} - 1 \cdot \frac{\cos nx}{-n^2}\right]_{-\pi}^{0} + \left[(\pi - x) \cdot \frac{\sin nx}{n} - (-1) \cdot \frac{\cos nx}{n^2}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right] + \left[\frac{\cos n\pi}{n^2} + \frac{1}{n^2}\right]\right) = \frac{2}{n^2 \pi}(1 - \cos n\pi)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+x)\sin nx\,dx + \int_{0}^{\pi}(\pi-x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left(\left[(\pi+x)\frac{\cos nx}{n} - 1\cdot\frac{\sin nx}{-n^2}\right]_{-\pi}^{0} + \left[(\pi-x)\frac{\cos nx}{n} - (-1)\cdot\frac{\sin nx}{-n^2}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{\pi}{n}\right] + \left[-\frac{\pi}{n}\right]\right) = 0$$

Hence the required fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}(1-\cos n\pi)\cos nx$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$$

<u>24</u>
$$f(x) = \begin{cases} -4x & \text{if } -\pi < x \le 0 \\ 4x & \text{if } 0 \le x \le \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -4x\,dx + \int_{0}^{\pi} 4x\,dx\right) = \frac{1}{2\pi}\left[\left(-2x^2\right)_{-\pi}^{0} + \left(2x^2\right)_{0}^{\pi}\right]$$

$$\frac{1}{2\pi}\left((0+2\pi^2) + (2\pi^2 - 0)\right) = 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-4x)\cos nx\,dx + \int_{0}^{\pi}(4x)\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[(-4x)\cdot\frac{\sin nx}{n} - (-4)\frac{\cos nx}{-n^2}\right]_{-\pi}^{0} + \left[(4x)\frac{\sin nx}{n} - (4)\frac{\cos nx}{-n^2}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\left(\frac{-4}{n^2} + \frac{4}{n^2}\cos n\pi\right) + \left(\frac{4}{n^2}\cos n\pi - \frac{4}{n^2}\right)\right)$$

$$= \frac{8}{n^2\pi}(\cos n\pi - 1)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\cdot\sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-4x)\sin nx\,dx + \int_{0}^{\pi}(4x)\sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[(-4x)\frac{\cos nx}{-n} - (-4)\frac{\sin nx}{-n^2}\right]_{-\pi}^{0} + \left[(4x)\frac{\cos nx}{-n} - (4)\frac{\sin nx}{-n^2}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{4\pi}{n}\cos n\pi - \frac{4\pi}{n}\cos n\pi\right) = 0$$

Hence the required Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx = 2\pi + \sum_{n=1}^{\infty}\frac{8}{n^2\pi}(\cos n\pi - 1)\cos nx$$

$$= 2\pi + \left(\frac{-16}{\pi}\right)\cos x + \left(\frac{-16}{9\pi}\right)\cos 3x + \cdots \quad \text{(Ans)}$$

— x —

[11·2] ① Functions of any period $p = 2L$

Arbitrary point

The fourier series of the funⁿ $f(x)$ of period $2L$

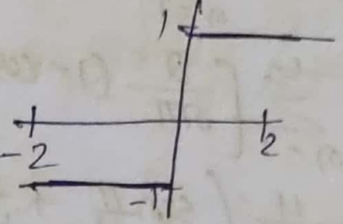
$$f(x) = a_0 + \sum\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

with Fourier co-efficients of $f(x)$ given by

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\cos\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$



Q.8-17

Exercise-11.2 Fourier Ser for per $p = 2L$, odd or even

Is the fn then even or odd, Find the Fourier Ser.

Q.(1) $f(x) = \begin{cases} -1 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$; $p = 2L = 4$

$p = 4$, $2L = 4$, $\Rightarrow L = 2$

Soln

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{4}\int_{-2}^{2} f(x)\,dx$$

$L = 2$,

$$= \frac{1}{4}\left(\int_{-2}^{0}(-1)\,dx + \int_{0}^{2}(1)\,dx\right)$$

$$= \frac{1}{4}\left[[-x]_{-2}^{0} + [x]_{0}^{2}\right] = \frac{1}{4}[(0-2)+(2-0)] = 0$$

Q.A

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\left(\int_{-2}^{0}(-1)\cos\frac{n\pi}{2}x\,dx + \int_{0}^{2}(1)\cdot\cos\frac{n\pi}{2}x\,dx\right)$$

$$= \frac{1}{2}\left(\left[\frac{(-1)\sin\left(\frac{n\pi}{2}\right)x}{\left(\frac{n\pi}{2}\right)}\right]_{-2}^{0} + \left[\frac{\sin\left(\frac{n\pi}{2}\right)x}{\left(\frac{n\pi}{2}\right)}\right]_{0}^{2}\right)$$

$$= \frac{1}{2}\left(-\frac{2}{n\pi}(0) + \frac{2}{n\pi}(0)\right) = 0$$

$u = \frac{n\pi}{2}x$
$du = \frac{n\pi}{2}dx$
$dx = \frac{2}{n\pi}du$
$\int \sin\frac{n\pi}{2}x\,dx$
$= \int \sin u \cdot \frac{2}{n\pi}du$
$= \frac{2}{n\pi}\int \sin u\,du$
$= \frac{2}{n\pi}(-\cos u)$
$= \frac{-2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{1}{2}\left(\int_{-2}^{0}(-1)\sin\frac{n\pi}{2}x\,dx + \int_{0}^{2}(1)\cdot\sin\frac{n\pi}{2}x\,dx\right)$$

$$= \frac{1}{2}\left(\left[\frac{\cos(\frac{n\pi}{2})x}{(\frac{n\pi}{2})}\right]_{-2}^{0} + \left[\frac{\cos(\frac{n\pi}{2})x}{-(\frac{n\pi}{2})}\right]_{0}^{2}\right)$$

$$= \frac{1}{2}\left(\frac{2}{n\pi}(1-\cos n\pi) - \frac{2}{n\pi}(\cos n\pi - 1)\right)$$

$$= \frac{2}{n\pi}(1-\cos n\pi)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$= 0 + \sum_{n=1}^{\infty}\left(0 + \frac{2}{n\pi}(1-\cos n\pi)\sin\frac{\pi n}{2}x\right)$$

$$= \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}(1-\cos n\pi)\sin\frac{n\pi}{2}x\right]$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \cdots\right)$$

Q. 11 $f(x) = \begin{cases} -x & (-1<x<0) \\ x & (0<x<1) \\ 1 & 1<x<3 \end{cases}$ $p = 4 = 2L$

Sol^n

$L = 2$

$$a_0 = \frac{1}{2L}\int_{-\frac{L}{2}}^{3L/2} f(x)\,dx = \frac{1}{4}\int_{-1}^{3} f(x)\,dx$$

$$= \frac{1}{4}\left(\int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx + \int_{1}^{3} 1\cdot dx\right)$$

$$= \frac{1}{4}\left(\left[\frac{-x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} + [x]_1^3\right)$$

$$= \frac{1}{4}\left(\left(0+\frac{1}{2}\right) + \left(\frac{1}{2}-0\right) + 2\right) = \frac{1}{4}(3) = \frac{3}{4}$$

$$a_n = \frac{1}{2L}\int_{-L/2}^{3L/2} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-1}^{3} f(x)\cos\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\left(\int_{-1}^{0}(-x)\cos\frac{n\pi}{2}x\,dx + \int_{0}^{1} x\cos\frac{n\pi}{2}x\,dx + \int_{1}^{3}\cos\frac{n\pi}{2}x\,dx\right)$$

$$= \frac{1}{2}\left(\left[(-x)\cdot\frac{\sin\frac{n\pi}{2}x}{(\frac{n\pi}{2})} - (-1)\cdot\frac{\cos(\frac{n\pi}{2})x}{(-\frac{n^2\pi^2}{4})}\right]_{-1}^{0} + \left[x\cdot\frac{\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}} - (1)\frac{\cos\frac{n\pi}{2}x}{(-\frac{n^2\pi^2}{4})}\right]_0^1\right.$$

$$\left. + \left[\frac{\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right]_1^3\right.$$

$$= \frac{1}{2}\left[\left(-\frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} + \frac{2}{n\pi}\sin\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\right) + \left(\frac{2}{n\pi}\sin\frac{3n\pi}{2} - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\right]$$

$$b_n = \frac{1}{L}\int_{-L/2}^{3L/2} f(x)\sin\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\int_{-1}^{3} f(x)\cdot\sin\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\left(\int_{-1}^{0}(-x)\sin\frac{n\pi}{2}x\,dx + \int_0^1 (x)\sin\frac{n\pi}{2}x\,dx + \int_1^3 \sin\frac{n\pi}{2}x\,dx\right)$$

$$= \frac{1}{2}\left(\left[(-x)\frac{\cos\frac{n\pi}{2}x}{-\frac{n\pi}{2}} - (-1)\frac{\sin\frac{n\pi}{2}x}{-\frac{n^2\pi^2}{4}}\right]_{-1}^{0} + \left[(x)\frac{\cos\frac{n\pi}{2}x}{-\frac{n\pi}{2}} - (1)\frac{\sin\frac{n\pi}{2}x}{-\frac{n^2\pi^2}{4}}\right]_0^1 + \left[\frac{\cos\frac{n\pi}{2}x}{-\frac{n\pi}{2}}\right]_1^3\right)$$

$$= \frac{1}{2}\left(\left(\frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) + \left(-\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) - \frac{2}{n\pi}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right).$$

$$= 0$$

Hence the required fouriere series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{L}x + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{L}x$$

$$= \frac{3}{4} - \frac{4}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{2}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2} + \cdots\right)$$

— x —

[12] Q3 (2) Even And Odd Functions
(Half-Range Expansion)

Even funⁿ :→
A funⁿ $f(x)$ is said to be even if $f(-x) = f(x), \forall x$.

Note
(1) The graph of $f(x)$ is symmetric about y-axis.
(2) $f(x)$ contains only even powers of x & may contain only $\cos x$, $\sec x$ & constant terms.
(3) $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$, when $f(x)$ is even.
(4) The sum of two even funⁿ is even.
(5) product of two even funⁿ is even.

Odd funⁿ :→
A funⁿ $f(x)$ is said to be odd iff $f(-x) = -f(x)\ \forall x$.

Notes :→
(1) The graph of $f(x)$ is symmetric about the origin.
(2) $f(x)$ contains only odd powers of x & may contain only $\sin x$, $\operatorname{cosec} x$, $\tan x$, $\cot x$.
(3) $\int_{-L}^{L} f(x)\,dx = 0$, when $f(x)$ is odd.
(4) The sum of two odd funⁿ is odd.
(5) product of an odd funⁿ & even funⁿ is odd.
(6) product of two odd funⁿ is even.

Fourier Series for Even & Odd functions :→

Case ① The fourier series of an even funⁿ of period $2L$ is a "Fourier Cosine series".

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi}{L}x\,dx$$

Case-(2) The Fourier series of odd fun of period 2L is a Fourier sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$b_n = \frac{2}{L} \int_0^{L} f(x) \cdot \sin \frac{n\pi}{L} x \, dx$$

(3) ~~Half~~ Half Range Expansion :→

A half range expansion containing only cosine terms is known as half-range Fourier cosine series of f(x) in the interval (0, L).

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right) x$$

$$a_0 = \frac{1}{L} \int_0^{L} f(x)\,dx \;, \quad a_n = \frac{2}{L} \int_0^{L} f(x) \cos \frac{n\pi}{L} x \, dx$$

Similarly half-range Fourier sine series contain only sine terms ;

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x$$

$$b_n = \frac{2}{L} \int_0^{L} f(x) \sin \frac{n\pi}{L} x \, dx.$$

Exercise : [11.2] Are the fol. fun even or odd or neither even nor odd.

Q.1 (a) $f(x) = |x|$.

$f(-x) = |(-x)| = |x| = f(x)$

$f(-x) = f(x)$ ∴ It is an even fun.

(b) $f(x) = x^2 \sin nx$.

$f(-x) = (-x)^2 \sin n(-x) = -x^2 \sin nx = -f(x)$

$f(-x) = -f(x)$ ∴ It is an odd fun.

(f) $f(x) = x \cosh x$

$f(-x) = -x \cosh(-x) = -x\cosh x = -f(x)$

$\Rightarrow f(-x) = -f(x)$ ∴ It is odd fun.

2 (a) $f(x) = \sin(x^2)$

$f(-x) = \sin(-x)^2 = \sin x^2 = f(x)$

$f(-x) = f(x)$ $\therefore$ It is an even fun^n.

(e) $f(x) = e^{\pi x}$

$f(-x) = e^{\pi(-x)} = e^{-\pi x}$

$f(x) \neq f(x)$ & $f(-x) \neq f(x)$

It is neighter even nor odd.

$e^x, e^{-|x|}$, $x^3\cos x$

Q.2 $\sin^2 x$, $\sin x$, $f_n x$

Q.3

$f(x) = x^3 \ (-\pi < x < \pi)$, $P = 2L = 2\pi$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 dx = \frac{1}{2\pi}\left[\frac{x^4}{4}\right]_{-\pi}^{\pi} = 0$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$

$= \frac{1}{\pi}\int_{-L}^{L} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3\cos nx\,dx = \frac{1}{\pi}\left[x^3\frac{\sin nx}{n} - 3x^2\frac{\cos nx}{-n^2} + 6x\frac{\sin nx}{-n^3} - 6\frac{\cos x}{n^4}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\left(\frac{3\pi^2}{n^2}\cos n\pi - \frac{6}{n^4}\cos n\pi\right) - \left(\frac{3\pi^2}{n^2}\cos n\pi - \frac{6}{n^4}\cos n\pi\right)\right]$

$= 0$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3\sin nx\,dx$

$= \frac{1}{\pi}\left[x^3\frac{\cos nx}{-n} - 3x^2\cdot\frac{\sin nx}{-n^2} + 6x\frac{\cos nx}{n^3} - 6\frac{\sin nx}{n^4}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left(\left(\frac{-\pi^3}{n}\cos n\pi + \frac{6\pi}{n^3}\cos n\pi\right) - \left(\frac{\pi^3}{n}\cos n\pi - \frac{6\pi}{n^3}\cos n\pi\right)\right)$

$= \frac{1}{\pi}\left(\frac{12\pi}{n^3}\cos n\pi - \frac{2\pi^3}{n}\cos n\pi\right)$

$= \frac{12}{n^3}\cos n\pi - \frac{2\pi^2}{n}\cos n\pi$

$L(f(t))$. F.S

$= \ln\left(\frac{s}{s-1}\right)$

$f(s) =$

$f(t) = L^{-1}\left[\ln\left(\frac{s}{s-1}\right)\right]$

$= L^{-1}(\ln s - \ln(s-1))$

Here $a_0 = 0$, $a_n = 0$, $b_n \neq 0$

It is an odd funⁿ in the given interval.

Q.5 $\quad f(x) = e^{-4x} \quad (-\pi < x < \pi)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-4x}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-4x}}{-4}\right]_{-\pi}^{\pi} = \frac{-1}{8\pi}\left(e^{-4\pi} - e^{4\pi}\right)$$

$$= \frac{-1}{8\pi}\left[(\cosh 4\pi - \sinh 4\pi) - (\cosh 4\pi + \sinh 4\pi)\right]$$

$$= 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-4x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-4x}}{16+n^2}\left(-4\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-4\pi}}{16+n^2}\left(-4\cos n\pi\right) - \frac{e^{4\pi}}{16+n^2}\left(-4\cos n\pi\right)\right]$$

$$= \frac{-4\cos n\pi}{\pi(16+n^2)}\left(e^{-4\pi} - e^{4\pi}\right) = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-4x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-4x}}{16+n^2}\left(-4\sin nx - n\cos nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-4\pi}}{16+n^2}\left(-n\cos n\pi\right) - \frac{e^{4\pi}}{16+n^2}\left(-n\cos n\pi\right)\right]$$

$$= 0$$

Here $a_0 = 0$, $a_n = 0$, $b_n = 0$

It is neither even nor odd in the given interval.

Q.6 $\quad f(x) = x^3 \sin x \quad (-\pi < x < \pi)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 \sin nx\,dx$$

$$= \frac{1}{2\pi}\left[x^3\frac{\cos nx}{-n} - 3x^2\frac{\sin nx}{-n^2} + 6x\frac{\cos nx}{n^3} - 6\frac{\sin nx}{n^4}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{-\pi^3}{n}\cos n\pi + \frac{6\pi}{n^3}\cos n\pi\right) - \left(\frac{\pi^3}{n}\cos n\pi - \frac{6\pi}{n^3}\cos n\pi\right)\right]$$

$$= \left(\frac{-\pi^2}{n} + \frac{6\pi}{n^3}\right)\cos n\pi$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3(\sin(1+n)x + \sin(1-n)x)\,dx$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} x^3\sin(1+n)x\,dx + \int_{-\pi}^{\pi} x^3\sin(1-n)x\,dx\right)$$

$$= \frac{1}{2\pi}\left(\left[x^3\frac{\cos(1+n)x}{-(1+n)} - 3x^2\frac{\sin(1+n)x}{-(1+n)^2} + 6x\frac{\cos(1+n)x}{(1+n)^3} - 6\frac{\sin(1+n)x}{(1+n)^4}\right]_{-\pi}^{\pi}\right.$$

$$\left. + \left[x^3\frac{\cos(1-n)x}{-(1-n)} - 3x^2\frac{\sin(1-n)x}{-(1-n)^2} + 6x\frac{\cos(1-n)x}{(1-n)^3} - 6\frac{\sin(1-n)x}{(1-n)^4}\right]_{-\pi}^{\pi}\right)$$

$$= \frac{1}{2\pi}\left[\left(\frac{-2\pi^3}{(1+n)}\cos(1+n)\pi + \frac{12\pi}{(1+n)^3}\cos(1+n)\pi\right) + \left(\frac{-2\pi^3}{1-n}\cos(1-n)\pi + \frac{12\pi}{(1-n)^3}\cos(1-n)\pi\right)\right]$$

$$b_n = \frac{1}{L}\int_{-L}^{L}\left(f(x)\cdot\sin\frac{n\pi}{L}x\,dx\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3\sin x\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3(\cos(1-n)x - \cos(1+n)x)\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} x^3\cos(1-n)x\,dx - \int_{-\pi}^{\pi} x^3\cos(1+n)x\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[x^3\frac{\sin(1-n)x}{(1-n)} - 3x^2\frac{\cos(1-n)x}{-(1-n)^2} + 6x\frac{\sin(1-n)x}{-(1-n)^3} - 6\frac{\cos(1-n)x}{(1-n)^4}\right]_{-\pi}^{\pi}\right.$$

$$\left. + \left[x^3\frac{\sin(1+n)x}{(1+n)} - 3x^2\frac{\cos(1+n)x}{-(1+n)^2} + 6x\frac{\sin(1+n)x}{-(1+n)^3} - 6\frac{\cos(1+n)x}{(1+n)^4}\right]_{-\pi}^{\pi}\right)$$

$$= 0$$

Here $a_0 \neq 0$, $a_n \neq 0$ but $b_n = 0$
Hence it is an even fun^n in the given interval.

Q.14 $f(x) = \begin{cases} \pi e^{-x} & \text{if } -\pi < x < 0 \\ \pi e^{x} & \text{if } 0 < x < \pi \end{cases}$

Sol^n

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0}\pi e^{-x}dx + \int_0^{\pi}\pi e^{x}dx\right)$$

$$= \frac{1}{2}\left(\left[-e^{-x}\right]_{-\pi}^{0} + \left[e^{x}\right]_0^{\pi}\right) = \frac{1}{2}\left((-1+e^{\pi}) + (e^{\pi}-1)\right) = e^{\pi}-1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0}\pi e^{-x}\cos nx\,dx + \int_0^{\pi}\pi e^{x}\cos nx\,dx\right)$$

$$= \left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_{-\pi}^{0} + \left[\frac{e^{x}}{1+n^2}(\cos nx + n\sin nx)\right]_0^{\pi}$$

$$= \left(\frac{-1}{1+n^2} + \frac{e^{\pi}\cos n\pi}{1+n^2}\right) + \left(\frac{e^{\pi}\cos n\pi}{1+n^2} - \frac{1}{1+n^2}\right)$$

$$\Rightarrow a_n = \frac{2(e^{\pi}\cos n\pi - 1)}{1+n^2}$$

$$a_1 = (-e^{\pi}-1),\ a_2 = 0,\ a_3 = \frac{1}{5}(-e^{\pi}-1)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0}\pi e^{-x}\sin nx\,dx + \int_0^{\pi}\pi e^{x}\sin nx\,dx\right)$$

$$= \left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_{-\pi}^{0} + \left[\frac{e^{x}}{1+n^2}(\sin nx - n\cos nx)\right]_0^{\pi}$$

$$= \left[\frac{(-n)}{1+n^2} + \frac{e^{\pi}n\cos n\pi}{1+n^2}\right] + \left[\frac{-ne^{\pi}\cos n\pi}{1+n^2} + \frac{n}{1+n^2}\right] = 0$$

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Hence the required fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$= (e^{\pi}-1) + (-e^{\pi}-1)\cos x + \frac{1}{5}(-e^{\pi}-1)\cos 3x + \cdots$$

Here $a_0 \neq 0$, $a_n \neq 0$, $b_n = 0$, It is an even fun^n in the given interval.

Q.23-29
For (a) Fourier cosine series
(b) Fourier sine series

23 ; $L = 4$

(a) ; $\frac{4}{\pi}$ $L = \pi$

25

29 $f(x)$: $\sin x$

17 $f(x) = 1 \quad [\because 0 < x < 2]$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{2}\int_0^2 dx = 1$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx = \int_0^2 \cos\frac{n\pi}{2}x\,dx$$

$$= \left[\frac{\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right]_0^2 = \frac{2}{n\pi}(\sin n\pi - 0) = 0$$

Hence the Half range cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x = 1$$

Again $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = \int_0^2 \sin\frac{n\pi}{2}x\,dx$$

$$= \left[\frac{\cos\frac{n\pi}{2}x}{-\frac{n\pi}{2}}\right]_0^2 = \frac{2}{n\pi}(1-\cos n\pi)$$

Hence the half-range fourier sine series;

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{2}x = \sum_{n=1}^{\infty}\frac{2}{n\pi}(1-\cos n\pi)\sin\frac{n\pi}{2}x$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \cdots\right)$$

21 $$f(x) = \begin{cases} 1 & (0<x<1) \\ 2 & (1<x<2) \end{cases}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\left(\int_0^1 dx + \int_1^2 2\,dx\right)$$

$$= \frac{1}{2}\left([x]_0^1 + [2x]_1^2\right) = \frac{1}{2}(1+2) = \frac{3}{2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx = \int_0^2 f(x)\cdot\cos\frac{n\pi}{2}x\,dx$$

$$= \int_0^1 \cos\frac{n\pi}{2}x\,dx + \int_1^2 2\cos\frac{n\pi}{2}x\,dx$$

$$= \left[\frac{\sin(\frac{n\pi}{2})x}{\frac{n\pi}{2}}\right]_0^1 + 2\left[\frac{\sin(\frac{n\pi}{2})x}{(\frac{n\pi}{2})}\right]_1^2$$

$$= \frac{2}{n\pi}\left(\sin\frac{n\pi}{2}\right) + \left(-\frac{4}{n\pi}\sin\frac{n\pi}{2}\right) = \frac{-2}{n\pi}\sin\frac{n\pi}{2}$$

Half range fourier cosine series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{L}x = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{-2}{n\pi}\sin\frac{\pi n}{2}\cdot\cos\frac{n\pi}{2}x$$

$$= \frac{3}{2} - \frac{2}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \cdots\right)$$

Again $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

$$= \int_0^2 f(x)\sin\frac{n\pi}{2}x\,dx = \int_0^1 \sin\frac{n\pi}{2}x\,dx + \int_1^2 2\sin\frac{n\pi}{2}x\,dx$$

$$= \left[\frac{\cos\frac{n\pi}{2}x}{-\frac{n\pi}{2}}\right]_0^1 + 2\left[\frac{\cos\frac{n\pi}{2}x}{-\frac{n\pi}{2}}\right]_1^2$$

$$= \frac{-2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right) - \frac{4}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$= \frac{2}{n\pi} - \frac{4}{n\pi}\cos n\pi + \frac{2}{n\pi}\cos\frac{n\pi}{2}$$

Hence half range fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{L}x = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1+\cos\frac{n\pi}{2} - 2\cos n\pi\right)\sin\frac{n\pi}{2}x$$

$$= \frac{6}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{3}\sin\pi x + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

—x—

[11.7]

Fourier Integral Theorem :→

A fun^n $f(x)$ which is piecewise continuous in every finite interval is absolutely integrable on the x-axis can be represented by a Fourier integral.

$$f(x) = \int_0^{\infty} [A(\omega) \cos\omega x + B(\omega) \sin\omega x] \, d\omega$$

where
$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos\omega v \, dv$$
$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin\omega v \, dv$$

Fourier Cosine Integral :→

When $f(x)$ is an even fun^n, $B(\omega) = 0$

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \cos\omega v \, dv$$

Here fourier cosine integral is

$$f(x) = \int_0^{\infty} A(\omega) \cos\omega x \, d\omega$$

Fourier Sine Integral :→

When $f(x)$ is an odd fun^n, $A(\omega) = 0$

$$B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \sin\omega v \, dv$$

Here fourier sine integral is

$$f(x) = \int_0^{\infty} B(\omega) \sin\omega x \, d\omega$$

Exercise

Q.1 Let
$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ e^{-x} & \text{if } x > 0 \end{cases}$$

Here
$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos\omega v \, dv$$
$$= \frac{1}{\pi} \left[\int_{-\infty}^{0} f(v) \cos\omega v \, dv + \int_0^{\infty} f(v) \cos\omega v \, dv \right]$$
$$= 0 + \frac{1}{\pi} \int_0^{\infty} e^{-v} \cos\omega v \, dv + 0$$
$$= \frac{1}{\pi} \left[\frac{e^{-v}}{1+\omega^2} (-\cos\omega v + \omega \sin\omega v) \right]_0^{\infty} = \frac{1}{\pi} \left(\frac{1}{1+\omega^2} \right)$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$
$$= \frac{1}{\pi}\left(\int_{-\infty}^{0} f(v)\sin\omega v\,dv + \int_{0}^{\infty} f(v)\sin\omega v\,dv\right)$$
$$= \frac{1}{\pi}\int_{0}^{\infty} e^{-v}\sin\omega v\,dv$$
$$= \frac{1}{\pi}\left[\frac{e^{-v}}{1+\omega^2}(-\sin\omega v - \omega\cos\omega v)\right]_0^{\infty} = \frac{\omega}{\pi(1+\omega^2)}$$

Fourier integral is

$$f(x) = \int_0^{\infty} A(\omega)\cos x\omega + B(\omega)\sin\omega x\,d\omega$$
$$= \int_0^{\infty}\left(\frac{1}{\pi(1+\omega^2)}\cos\omega x + \frac{\omega}{\pi(1+\omega^2)}\sin\omega x\right)d\omega$$
$$= \int_0^{\infty}\left(\frac{\cos\omega x + \omega\sin\omega x}{\pi(1+\omega^2)}\right)d\omega$$

If $f(x) = e^{-x}$; $x > 0$

$$\therefore \int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\,d\omega = \pi e^{-x}$$

If $f(x) = 0$, $x < 0$

$$\int_0^{\infty}\frac{\cos x\omega + \omega\sin\omega x}{1+\omega^2}\,d\omega = 0$$

At $x = 0$, $f(x)$ has discontinuity, so

$$f(x) = \frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2}(1+0) = \frac{1}{2}$$

If $f(x) = \frac{1}{2}$, if $x = 0$

$$\int_0^{\infty}\frac{\cos x\omega + \omega\sin x\omega}{1+\omega^2}\,d\omega = \frac{\pi}{2}$$

Q.6 Consider the funⁿ defined by

Q.4
7-12

$$f(x) = \begin{cases}\sin x & 0 \le x \le \pi\\ 0 & x > \pi\end{cases}$$

$$B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin\omega v\,dv$$
$$= \frac{2}{\pi}\int_0^{\pi}\sin v\cdot\sin\omega v\,dv$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\cos(1-w)v - \cos(1+w)v\right)dv$$

$$= \frac{1}{\pi}\left[\frac{\sin(1-w)v}{1-w} - \frac{\sin(1+w)v}{1+w}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(\pi-\pi w)}{1-w} - \frac{\sin(\pi+\pi w)}{1+w}\right]$$

$$= \frac{1}{\pi}\left(\frac{\sin\pi w}{1-w} + \frac{\sin\pi w}{1+w}\right)$$

$$= \frac{2}{\pi}\left(\frac{\sin\pi w}{1-w^2}\right)$$

Here Fourier sine integral is

$$f(x) = \int_0^{\infty} B(w)\sin wx\, dw$$

$$= \int_0^{\infty} \frac{2\sin\pi w}{\pi(1-w^2)} \sin wx\, dw$$

If $f(x) = \sin x \; (0 < x \leq \pi)$:

$$\sin x = \int_0^{\infty} \frac{2\sin\pi w}{\pi(1-w^2)} \sin wx\, dw$$

$$\Rightarrow \boxed{\int_0^{\infty} \frac{\sin\pi w}{(1-w^2)} \sin wx\, dx = \frac{\pi}{2}\sin x}$$

If $f(x) = 0 \;;\; x > \pi$

$$f(x) = \int_0^{\infty} \frac{2\sin\pi w}{\pi(1-w^2)} \sin wx\, dx$$

$$\Rightarrow 0 = \int_0^{\infty} \frac{2\sin\pi w}{\pi(1-w^2)} \sin wx\, dx$$

$$\Rightarrow \boxed{\int_0^{\infty} \frac{\sin\pi w}{(1-w^2)} \sin wx\, dx = 0}$$

Q.10 $$f(x) = \begin{cases} 1-\frac{x}{2} & , \text{if } 1 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$$

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos wv\, dv$$

$$= \frac{2}{\pi}\int_1^{2}\left(1-\frac{v}{2}\right)\cos wv\, dv$$

$$= \frac{2}{\pi}\left[\left(1-\frac{v}{2}\right)\frac{\sin wv}{w} - \left(\frac{-1}{2}\right)\frac{\cos wv}{-w^2}\right]_1^2$$

$$= \frac{2}{\pi}\left(\frac{-1}{2w^2}\cos 2w + \frac{1}{2w}\sin w + \frac{1}{2w^2}\cos w\right)$$

Hence Fourier cosine integral is

$$f(x) = \int_0^\infty A(w)\cos wx \, dw$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{-1}{2w^2}\cos 2w + \frac{1}{2w}\sin w + \frac{1}{2w^2}\cos w\right)\cos xw \, dw$$

17

$$f(x) = \begin{cases} \pi - x & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$$

$$B(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv \, dv = \frac{2}{\pi}\left[(\pi - v)\frac{\sin wv}{w} - (-1)\frac{\cos wv}{-w^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left(-\frac{\cos w\pi}{w^2} + \frac{1}{w^2}\right)$$

$$= \frac{2}{\pi w^2}(1-\cos \pi w)$$

Hence Fourier sine integral is

$$f(x) = \int_0^\infty B(w)\sin xw \, dw$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos \pi w}{w^2}\right)\sin wx \, dw$$

Q.19

$$f(x) = \begin{cases} a - x & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$$

$$B(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv \, dv$$

$$= \frac{2}{\pi}\int_0^a (a-v)\cos wv \, dv$$

$$= \frac{2}{\pi}\left[(a-v)\frac{\sin wv}{w} - (-1)\frac{\cos wv}{-w^2}\right]_0^a$$

$$= \frac{2}{\pi}\left(-\frac{\cos wa}{w^2} + \frac{1}{w^2}\right) = \frac{2}{\pi}w^2(1-\cos aw)$$

Hence Fourier sine integral is

$$f(x) = \int_0^\infty B(w)\sin xw \, dw = \frac{2}{\pi}\int_0^\infty \frac{1-\cos aw}{w^2}\sin xw \, dw$$

11.7

[7-12] fourier cosine integrals; represent $f(x)$ as an integral;

7 $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

As $f(x) = \int_0^\infty A(w) \cos xw \, dw$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv$$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv = \frac{2}{\pi} \int_0^1 1 \cdot \cos wv \, dv$$

$$= \frac{2}{\pi} \left. \frac{\sin wv}{w} \right]_0^1 = \frac{2}{\pi w} (\sin w - 0) = \frac{2 \sin w}{\pi w}$$

Cosine integral is

$$f(x) = \int_0^\infty \frac{2}{\pi w} \sin w \cdot \cos xw \, dw$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin w \cdot \cos xw}{w} \, dw$$

8 $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv = \frac{2}{\pi} \int_0^1 v^2 \cos wv \, dv$$

$$= \frac{2}{\pi} \left[v^2 \cdot \frac{\sin wv}{w} - \int_0^1 2v \cdot \frac{\sin wv}{w} \, dv \right]$$

$$= \frac{2}{\pi} \left[v^2 \cdot \frac{\sin wv}{w} - \frac{2}{w} \left\{ v \cdot \frac{-\cos wv}{w} + \int_0^1 1 \cdot \frac{\cos wv}{w} \, dw \right\} \right]$$

$$= \frac{2}{\pi}\left[v^2 \cdot \frac{\sin wv}{w} + \frac{2v}{w^2}\cos wv + \frac{2\sin wv}{w^3} \right]_0^1$$

$$= \frac{2}{\pi}\left[\left(\frac{\sin w}{w} + \frac{2}{w^2}\cos w + \frac{2}{w^3}\sin w\right) - (0+0+0)\right]$$

$$= \frac{2}{\pi}\left(\frac{\sin w}{w} + \frac{2}{w^2}\cos w + \frac{2}{w^3}\sin w\right)$$

Hence fourier cosine integral is

$$f(x) = \int_0^\infty A(w)\cos wx \, dw$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{\sin w}{w} + \frac{2}{w^2}\cos w + \frac{2}{w^3}\sin w\right)\cos wx \, dw$$

10 $f(x) = \begin{cases} a^2 - x^2, & 0 < x < a \\ 0, & x > a \end{cases}$

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv \, dv = \frac{2}{\pi}\int_0^a (a^2 - v^2)\cos wv \, dv$$

$$= \frac{2}{\pi}\left[\int_0^a a^2 \cos wv \, dv - \int_0^a v^2 \cos wv \, dv\right]$$

$$= \frac{2}{\pi}\left[a^2 \frac{\sin wv}{w}\right]_0^a - \frac{2}{\pi}\left[v^2 \frac{\sin wv}{w} + \frac{2v}{w^2}\cos wv + \frac{2}{w^3}\sin wv\right]_0^a$$

$$= \frac{2}{\pi}\times\frac{a^2(\sin aw)}{w} - \frac{2}{\pi}\left[\frac{a^2}{w}\sin wa + \frac{2a}{a^2}\cos aw + \frac{2}{a^3}\sin aw\right]$$

$$= -\frac{4}{\pi a}\cos aw - \frac{4}{\pi a^3}\sin aw = -\frac{4}{\pi a}\left[\cos aw + \frac{\sin aw}{a^2}\right]$$

Hence fourier cosine integral is

$$f(x) = \int_0^\infty A(w)\cos wx \, dw$$

$$= \frac{-4}{\pi a}\int_0^\infty \left(\cos aw + \frac{\sin aw}{a^2}\right)\cos wx \, dw$$

12

$f(x) = \begin{cases} \sin x ; & 0 < x < \pi \\ 0 ; & x > \pi \end{cases}$

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv \, dv = \frac{2}{\pi}\int_0^\pi \sin v \cdot \cos wv \, dv$$

$$= \frac{2}{\pi}\int_0^\pi \frac{1}{2}(\sin(v + wv) + \sin(v - wv)) \, dv$$

$$= \frac{1}{\pi}\left[\int_0^\pi \sin(1+w)v \, dv + \int_0^\pi \sin(1-w)v \, dv\right]$$

$$= \frac{2}{\pi}\left(\frac{1+\cos\pi w}{1-w^2}\right)$$

[11.7]

[16-20] Fourier sine integral ;

16 $f(x) = \begin{cases} x, & 0 < x < a \\ 0, & x > a \end{cases}$

Sine int.

$f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega$; $B(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \sin \omega v \, dv$

$B(\omega) = \frac{2}{\pi} \int_0^a v \cdot \sin \omega v \, dv = \frac{2}{\pi} \left[v \cdot \frac{-\cos \omega v}{\omega} + \frac{\sin \omega v}{\omega^2} \right]_0^a$

$= \frac{2}{\pi} \left[-\frac{v \cos \omega v}{\omega} + \frac{\sin \omega v}{\omega^2} \right]_0^a$

$= \frac{2}{\pi} \left[-\frac{a \cos a\omega}{\omega} + \frac{\sin a\omega}{\omega^2} \right]$

By sine integral ;

$f(x) = \int_0^\infty \frac{2}{\pi} \left[-\frac{a}{\omega} \cos a\omega + \frac{\sin a\omega}{\omega^2} \right] . \sin \omega x \, d\omega$

17 $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

$B(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \cdot \sin \omega v \, dv = \frac{2}{\pi} \int_0^1 1 . \sin \omega v \, dv = \frac{2}{\pi} \left[-\frac{\cos \omega v}{\omega} \right]_0^1$

$= \frac{-2}{\pi \omega} (\cos \omega - 1) = \frac{2}{\pi \omega} (1 - \cos \omega)$

By sine integral ; $f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega$

$= \frac{2}{\pi} \int_0^\infty \frac{1}{\omega} (1 - \cos \omega) . \sin \omega x \, d\omega$

18 $f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$

$$B(w) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin wv\,dv = \frac{2}{\pi}\int_0^{\pi}\cos v\cdot\sin wv\,dv$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\left(\sin(v+wv)+\sin(v-wv)\right)dv$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}\sin(1+w)v\,dv + \int_0^{\pi}\sin(1-w)v\,dv\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{-\cos(1+w)v}{1+w}\right]_0^{\pi} + \left[-\frac{\cos(1-w)v}{1-w}\right]_0^{\pi}\right\}$$

$$= \frac{-1}{\pi}\left\{\left(\frac{\cos(1+w)\pi - 1}{1+w}\right) + \left(\frac{\cos(1-w)\pi - 1}{1-w}\right)\right\}$$

$$= \frac{-1}{\pi}\left[\frac{(1-w)\cos(1+w)\pi - 1 + w + (1+w)\cos(1-w)\pi - 1 - w}{1-w^2}\right]$$

$$= \frac{-1}{\pi}\left[\frac{(1-w)(\cos\pi\cdot\cos w\pi - \sin\pi\sin\pi w) + (1+w)(\cos\pi\cos\pi w + \sin\pi\sin\pi w) - 2}{1-w^2}\right]$$

$$= \frac{-1}{\pi}\left[\frac{-\cos w\pi + w\cos w\pi - \cos w\pi - w\cos\pi w - 2}{1-w^2}\right]$$

$$= \frac{2}{\pi}\left(\frac{\cos w\pi + 1}{1-w^2}\right)$$

1-6 Show that the integrals:

(1) $$\int_0^{\infty}\frac{\cos xw + w\sin xw}{1+w^2}\,dw = \begin{cases} 0 & , x<0 \\ \pi/2 & , x=0 \\ \pi e^{-x} & , x>0 \end{cases}$$

R.H.S

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} f(v)\cos wv\,dv + \int_0^{\infty} f(v)\cos wv\,dv\right]$$

$$= \frac{1}{\pi}\int_0^{\infty}\pi e^{-v}\cos wv\,dv = \int_0^{\infty} e^{-v}\cos wv\,dv$$

$$= \left[\frac{e^{-v}}{1+w^2}(-\cos wv + w\sin wv)\right]_0^{\infty} = \frac{1}{1+w^2}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv = \frac{1}{\pi}\int_0^{\infty}\pi e^{-v}\sin wv\,dv = \frac{w}{1+w^2}$$

Fourier integral $f(x) = \int_0^{\infty}(A(w)\cos wx + B(w)\sin wx)\,dw$

$$= \int_0^{\infty}\left(\frac{1}{1+w^2}\cos wx + \frac{w}{1+w^2}\sin wx\right)dw = \frac{1}{1+w^2}\int_0^{\infty}(\cos wx + w\sin wx)\,dw$$

If $f(x) = e^{-x}, x>0$ $\quad \int_0^{\infty}\frac{\cos wx + w\sin wx}{1+w^2}\,dw = \pi e^{-x}$

If $f(x) = \pi/2, x=0$, $\quad = \pi/2$

[11.8] Fourier Cosine & sine Transforms

Fourier Cosine Transforms :–

The formula for fourier cosine transform is

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos\omega x \, dx$$

Fourier sine Transforms :→

The formula for fourier sine formula is

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin\omega x \, dx$$

Properties :→

(1) $F_c(af(x) + b\,g(x)) = a F_c f(x) + b F_c g(x)$

(2) $F_s(af(x) + b\,g(x)) = a F_s f(x) + b F_s g(x)$

Ex $f(x) = e^{-ax}$; $a>0$, find fourier cosine transform of $f(x)$.

Sol^n

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \underset{V}{e^{-ax}} \underset{U}{\cos\omega x} \, dx$$

(By parts) $$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+\omega^2} (-a\cos\omega x + \omega \sin\omega x) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+\omega^2} \right)$$

(or) $F_c(f''(x)) = -\omega^2 F_c(f(x)) - \sqrt{\frac{2}{\pi}} f'(0)$

$f(x) = e^{-ax}$, $f'(x) = -ae^{-ax}$, $f'(0) = -a$, $f''(x) = a^2 e^{-ax}$

$F_c(a^2 e^{-ax}) = -\omega^2 F_c(e^{-ax}) - \sqrt{\frac{2}{\pi}}(-a)$

$a^2 F_c(e^{-ax}) + \omega^2 F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\, a$,

$F_c(e^{-ax})(a^2+\omega^2) = \sqrt{\frac{2}{\pi}}\, a$

$F_c(e^{-ax}) = \dfrac{\sqrt{\frac{2}{\pi}}\, a}{a^2+\omega^2}$

$f(x) = \{ \quad \}$ $\vec{a}$ sum ; a^n t

11.8

L-S Fourier Cosine transform ; $f_c(\omega)$

[11.8] Fourier Cos & Sin trans

F.C.T. →
The formula for Fourier cosine transform is $F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos\omega x\,dx$

Fourier sine tra
The sine tr $F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin\omega x\,dx$

property
(1) $F_c(af(x) + bg(x)) = aF_c(f(x)) + bF_c(g(x))$
(2) $F_s(af(x) + bg(x)) = aF_s(f(x)) + bF_s(g(x))$

(1) find the cosine transform

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 1\cdot\cos\omega x\,dx + \int_1^2 -1\cdot\cos\omega x\,dx + \int_2^\infty 0\cdot\cos\omega x\,dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{\sin\omega x}{\omega}\right]_0^1 - \left[\frac{\sin\omega x}{\omega}\right]_1^2 \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin\omega}{\omega} - \left(\frac{\sin 2\omega}{\omega} - \frac{\sin\omega}{\omega}\right) \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2\sin\omega - \sin 2\omega}{\omega} \right)$$

(2)
$$f(x) = \begin{cases} x ; & 0 < x < 2 \\ 0 ; & x > 2 \end{cases}$$

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \left[\int_0^2 x\cdot\cos\omega x\,dx + 0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[x\cdot\frac{\sin\omega x}{\omega} + \frac{\cos\omega x}{\omega} \right]_0^2$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2}{\omega}\sin 2\omega + \frac{\cos 2\omega}{\omega} + 0 - \frac{1}{\omega} \right]$$

$$= \sqrt{\frac{2}{\pi\omega}} \left[2\sin 2\omega + \cos 2\omega - 1 \right]$$

(5)
~~4.~~ $\cancel{f(x) = x^{!}\sin x \; ; \; 0 < x < \infty}$

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x}\sin x \cdot$$

(Q-15) (5)
$$f(x) = \begin{cases} x^2 ; & 0 < x < 1 \\ 0 ; & x > 1 \end{cases}$$

(6)
$$g(x) = \begin{cases} 2 ; & 0 < x < 1 \\ 0 ; & x > 1 \end{cases}$$

Fourier sine transform ;

9 find $F_s(e^{-ax})$, $a>0$ find by

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \underset{V}{e^{-ax}} \cdot \underset{U}{\sin \omega x \, dx} \right]$$

$$\cancel{\sqrt{\frac{2}{\pi}} \left[e^{-ax} \left(\frac{-\cos \omega x}{\omega}\right) - \int_0^\infty -a e^{-ax} \cdot \left(-\frac{\cos \omega x}{\omega}\right) dx \right]_0^\infty}$$

$$\cancel{= \sqrt{\frac{2}{\pi}} \left[\frac{-e^{-ax} \cos \omega x}{\omega} - \frac{a}{\omega} \int_0^\infty e^{-ax} \cdot \cos \omega x \, dx \right]}$$

$$= \sqrt{\frac{2}{\pi}} \left[\sin \omega x \cdot \frac{e^{-ax}}{-a} - \int_0^\infty \omega \cos \omega x \cdot \frac{e^{-ax}}{-a} \right.$$

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cdot \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\sin \omega x \cdot \left(-\frac{e^{-ax}}{a}\right) - \int_0^\infty \omega \cdot \cos \omega x \cdot \frac{e^{-ax}}{-a} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{ -\frac{\sin \omega x \cdot e^{-ax}}{a} + \frac{\omega}{a} \left[\int_0^\infty \cos \omega x \cdot e^{-ax} \right] \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ -\frac{\sin \omega x \cdot e^{-ax}}{a} + \frac{\omega}{a} \left[\cos \omega x \cdot \frac{e^{-ax}}{-a} - \int_0^\infty -\omega \sin \omega x \cdot \frac{e^{-ax}}{-a} \right] \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ -\frac{\sin \omega x \cdot e^{-ax}}{a} - \frac{\omega}{a^2} \cos \omega x \cdot e^{-ax} - \frac{\omega^2}{a^2} \int_0^\infty \sin \omega x \cdot e^{-ax} \right\}$$

$$\Rightarrow \left(1 + \frac{\omega^2}{a^2}\right) \int_0^\infty e^{-ax} \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} \left\{ -\frac{\sin \omega x \cdot e^{-ax}}{a} - \frac{\omega}{a^2} \cos \omega x \cdot e^{-ax} \right\}_0^\infty$$

$$\Rightarrow \int_0^\infty e^{-ax} \cdot \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} \times \frac{a^2}{a^2+\omega^2} \left\{ \frac{-a \sin \omega x \cdot e^{-ax} - \omega \cos \omega x \cdot e^{-ax}}{a^2} \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+\omega^2} \left(-a \sin \omega x - \omega \cos \omega x \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi (a^2+\omega^2)}} \left[0 - (0 - \omega \cdot 1) \right]$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\omega}{a^2+\omega^2}$$

11

$$f(x) = \begin{cases} x^2 ; & 0 < x < 1 \\ 0 ; & x > 1 \end{cases}$$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x^2 \sin\omega x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[x^2 \cdot \frac{\cos\omega x}{(-\omega)} - \int_0^1 2x \cdot \frac{\cos\omega x}{(-\omega)} dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{x^2 \cdot \cos\omega x}{\omega} + \frac{2}{\omega} \left\{ \int_0^1 x \cdot \cos\omega x \, dx \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{x^2 \cdot \cos\omega x}{\omega} + \frac{2}{\omega} \left\{ x \cdot \frac{\sin\omega x}{\omega} - \int_0^1 \frac{\sin\omega x}{\omega} dx \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{x^2 \cos\omega x}{\omega} + \frac{2x \cdot \sin\omega x}{\omega^2} + \frac{2}{\omega^3} \cos\omega x \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left(\frac{-\cos\omega}{\omega} + \frac{2\sin\omega}{\omega^2} + \frac{2}{\omega^3} \cos\omega \right) - \left(0 + 0 + \frac{2}{\omega^3} \right) \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos\omega}{\omega} + \frac{2\sin\omega}{\omega^2} + \frac{2}{\omega^3} \cos\omega - \frac{2}{\omega^3} \right]$$

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Q.1 $f(x) = \begin{cases} -1 & \text{if } 0<x<1 \\ 1 & \text{if } 1<x<2 \\ 0 & \text{if } x>2 \end{cases}$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx = \sqrt{\frac{2}{\pi}}\left[\int_0^1 f(x)\cos wx\,dx + \int_1^2 f(x)\cos wx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\int_0^1 (-1)\cos wx\,dx + \int_1^2 (1)\cos wx\,dx\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\left[-\frac{\sin wx}{w}\right]_0^1 + \left[\frac{\sin wx}{w}\right]_1^2\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(-\frac{\sin w}{w} + \frac{\sin 2w}{w} - \frac{\sin w}{w}\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{\sin 2w - 2\sin w}{w}\right)$$

Q.6 $F_c^{-1}(e^{-w})$ by integration.

$$f(x) = F_c^{-1}(e^{-w}) = \sqrt{\frac{2}{\pi}}\int_0^\infty \underset{v}{e^{-w}}\,\underset{u}{\cos xw}\,dw$$

By parts method
$\int u\,v\,dw$
$= u\int v\,dw - \int\left(\frac{du}{dw}\cdot\int v\,dw\right)dw$
$= \cos xw \int e^{-w}dw + \int\left(x\sin xw \cdot \frac{e^{-w}}{-1}\right)dw$
$= \cos xw\cdot\frac{e^{-w}}{-1} + \ldots$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{e^{-w}}{1+x^2}(-\cos xw + x\sin xw)\right)_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[(0) + \frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+x^2}\right)$$

Q.5 obtain $F_c^{-1}\left(\frac{1}{1+w^2}\right)$

$$f(x) = F_c^{-1}\left(\frac{1}{1+w^2}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+w^2}\cos xw\,dw$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos xw}{1+w^2}\,dw = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}e^{-x}\right) \text{ if } x>0$$

$$= \sqrt{\frac{\pi}{2}}\,e^{-x} \text{ if } x>0$$

Q.11 $F_s(e^{-\pi x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\pi x}\sin wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-\pi x}}{\pi^2+w^2}(-\pi\sin wx - w\cos wx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{w}{\pi^2+w^2}\right)$$

Q.14

Let $f(x) = \sin x$ if $0<x<\pi$, 0 if $x>\pi$; find $F_s(f(x))$

Sol^n $F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^{\pi} \sin x \cdot \sin wx \, dx$

$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{\pi} (\cos(1-w)x - \cos(1+w)x) \, dx$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1-w)x}{1-w} - \frac{\sin(1+w)x}{1+w} \right]_0^{\pi}$

$= \frac{1}{\sqrt{2\pi}} \left(\frac{\sin(1-w)\pi}{1-w} - \frac{\sin(1+w)\pi}{1+w} \right)$

$= \frac{1}{\sqrt{2\pi}} \left(\frac{2\sin w\pi}{1-w^2} \right)$

$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin w\pi}{1-w^2} \right)$ (Ans)

Q.19 Show that $f(ax)$ has the fourier sine transform $\frac{1}{a} F_s(w/a)$

Sol^n $F_s(f(ax)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(ax) \sin wx \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \frac{w}{a} t \, \frac{dt}{a}$

$= \frac{1}{a}\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \frac{w}{a} t \, dt$

$= \frac{1}{a} F_s(w/a)$

— x —

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[11.9] Fourier Transform

The fourier integral is

$$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\,dw$$

$$\left[\text{where } A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv\right]$$

$$= \int_0^{\infty}\left[\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv\right)\cos wx + \left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv\right)\sin wx\right]dw$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)\left[(\cos wv\cos wx + \sin wv \sin wx)\,dv\right]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\,dv\right]dw \quad \text{------ (1)}$$

We claim that the integral of the form (1) with 'sin' instead of 'cos' is zero.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\,dv\right]dw = 0 \quad \text{------ (2)}$$

Adding (1) & (2)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\,e^{iw(x-v)}\,dv\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\,e^{-iwv}\,dv\right]e^{iwx}\,dx$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\,e^{iwx}\,dx$$

where $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\,e^{-iwv}\,dv$

It is called as Fourier transform.

Existance of the Fourier Transform :→

1. $f(x)$ is piecewise cont. on every finite interval.
2. $f(x)$ is absolutely integrable on the x-axis.

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Exercise - 11.9

Q.1 $\frac{1}{i} = \frac{i}{i \cdot i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

we know $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

Adding we get $e^{ix} + e^{-ix} = 2\cos x$

substracting we get $e^{ix} - e^{-ix} = 2i\sin x$

Q.2 $f(x) = \begin{cases} e^{kx} & \text{if } x<0 \\ 0 & \text{if } x>0 \end{cases} \quad (\because k>0)$

Sol $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{kx} e^{-i\omega x} dx + 0$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(k-i\omega)x} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(k-i\omega)x}}{k-i\omega} \right]_{-\infty}^{0}$

$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{k-i\omega} \right)$

Q.3 $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{b} k e^{-i\omega x} dx$

$= \frac{k}{\sqrt{2\pi}} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{0}^{b} = \frac{k}{\sqrt{2\pi}} \left(\frac{e^{-i\omega b} - 1}{-i\omega} \right)$

$= \frac{ik(e^{-i\omega b} - 1)}{\sqrt{2\pi}\omega}$

Q.5 $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} k e^{-i\omega x} dx$

$= \frac{k}{\sqrt{2\pi}} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-1}^{1} = \frac{k}{\sqrt{2\pi}} \left(\frac{e^{-i\omega} - e^{i\omega}}{-i\omega} \right)$

$= \sqrt{\frac{2}{\pi}} \frac{k}{\omega} (\sin \omega)$ Ans)

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Q.6 $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x e^{-i\omega x}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{x e^{-i\omega x}}{-i\omega} - \frac{e^{-i\omega x}}{-\omega^2} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(-\frac{e^{-i\omega}}{i\omega} + \frac{e^{-i\omega}}{\omega^2} \right) - \left(\frac{e^{i\omega}}{i\omega} + \frac{e^{i\omega}}{\omega^2} \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{i\cos\omega}{\omega} - \frac{i\sin\omega}{\omega^2} \right)$$

Q.7 $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{1} x e^{-i\omega x}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[x \frac{e^{-i\omega x}}{-i\omega} - \frac{e^{-i\omega x}}{(-\omega^2)} \right]_{0}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left(-\frac{e^{-i\omega}}{i\omega} + \frac{e^{-i\omega}}{\omega^2} - \frac{1}{\omega^2} \right)$$

$$= \frac{1}{\omega^2\sqrt{2\pi}} \left[(1+i\omega)e^{-i\omega} - 1 \right]$$

Q.9 $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-1}^{0} -e^{-i\omega x}\, dx + \int_{0}^{1} e^{-i\omega x}\, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[\frac{e^{-i\omega x}}{i\omega} \right]_{-1}^{0} + \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{0}^{1} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{i\omega}(1 - e^{i\omega}) - \frac{1}{i\omega}(e^{-i\omega} - 1) \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{i\omega} - \left(\frac{e^{-i\omega} + e^{i\omega}}{i\omega} \right) \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{(\cos\omega - 1)}{\omega}$$ (Ans)

——— x ———

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$e^{-ix} = \cos x - i\sin x$

11.9

2-11 Fourier transform by integration :

2 $f(x) = \begin{cases} e^{2ix} & ; -1 < x < 1 \\ 0 & ; \text{otherwise} \end{cases}$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} f(x) e^{-i\omega x} dx + \int_{-1}^{1} f(x) e^{-i\omega x} dx + \int_{1}^{\infty} f(x) \cdot e^{-i\omega x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} f(x) \cdot e^{-i\omega x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} e^{2ix} \cdot e^{-i\omega x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i(2-\omega)x} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(2-\omega)x}}{(2-\omega)} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(2-\omega)}}{2-\omega} - \frac{e^{i(\omega+2)}}{2-\omega} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{1 \times 2i}{2-\omega} \left[\frac{e^{i(2-\omega)} - e^{-i(2-\omega)}}{2i} \right]$$

$$= \frac{2i}{\sqrt{2\pi}(2-\omega)} \times \sin(2-\omega)$$

3 $f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

4 $f(x) = \begin{cases} e^{kx} & ; x < 0 \\ 0 & ; x > 0 \end{cases} \quad k > 0$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{kx} \cdot e^{-i\omega x} dx + 0$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(k-i\omega)x} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(k-i\omega)x}}{k-i\omega} \right]_{-\infty}^{0}$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - \frac{1}{k-i\omega} \right) = \frac{1}{\sqrt{2\pi}(k-i\omega)}$$

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$$f(x) = \begin{cases} e^{x} ; & -a < x < a \\ 0 ; & \text{otherwise} \end{cases}$$

$$f(x) = e^{-|x|} \quad (-\infty < x < \infty)$$

$$f(x) = \begin{cases} x ; & 0 < x < a \\ 0 ; & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} x e^{-x} ; & -1 < x < 0 \\ 0 , & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} |x| ; & -1 < x < 1 \\ 0 ; & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} x ; & -1 < x < 1 \\ 0 ; & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} -1 ; & -1 < x < 0 \\ 1 ; & 0 < x < 1 \\ 0 ; & \text{otherwise} \end{cases}$$

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